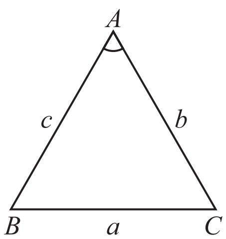
CHAPTER

25 Solutions of Triangles

1. Sine Rule:

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$



2. Cosine Formula:

(*i*) $\cos A=\frac{b^2+c^2-a^2}{2bc}$

(*ii*) $\cos B=\frac{c^2+a^2-b^2}{2ca}$

(*iii*) $\cos C=\frac{a^2+b^2-c^2}{2ab}$

3. Projection Formula

(*i*) $a=b\cos C+c\cos B$

(*ii*) $b=c\cos A+a\cos C$

(*iii*) $c=a\cos B+b\cos A$

4. Napier's Analogy - Tangent Rule

(*i*) $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$

(*ii*) $\tan\frac{C-A}{2}=\frac{c-a}{c+a}\cot\frac{B}{2}$

(*iii*) $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$

5. Trigonometric Functions of Half Angles

(*i*) $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$; $\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}}$;

$\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$

(*ii*) $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$; $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}}$; $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$

(*iii*) $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\frac{\Delta}{s(s-a)}$ where $s=\frac{a+b+c}{2}$ is semi perimetre of triangle.

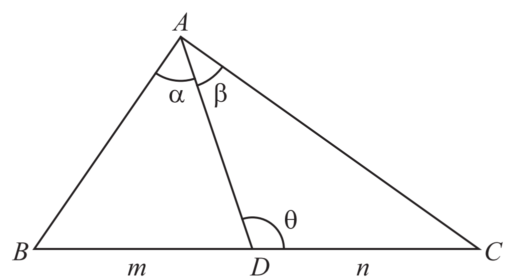
(*iv*) $\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}=\frac{2\Delta}{bc}$

6. Area of Triangle (Δ)

$\Delta=\frac{1}{2}ab\sin C=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B$

$=\sqrt{s(s-a)(s-b)(s-c)}$.

7. m-n Rule



If $BD:DC=m:n$, then

$(m+n)\cot\theta=m\cot\alpha-n\cot\beta$

$=n\cot B-m\cot C$

8. Radius of Circumcircle

$$R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}=\frac{abc}{4\Delta}$$

9. Radius of The Incircle

(*i*) $r=\frac{\Delta}{s}$

(*ii*) $r=(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2}$

(*iii*) $r=\frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$ and so on (*iv*) $r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

10. Radius of The Ex-Circles

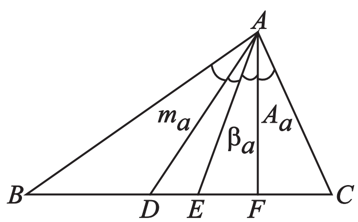
(*i*) $r_1 = \frac{\Delta}{s-a}; r_2 = \frac{\Delta}{s-b}; r_3 = \frac{\Delta}{s-c}$

(*ii*) $r_1 = s \tan \frac{A}{2}; r_2 = s \tan \frac{B}{2}; r_3 = s \tan \frac{C}{2}$

(*iii*) $r_1 = \frac{a \cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}}$ and so on.

(*iv*) $r_1 = 4R \sin \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2}$

11. Length of Angle Bisectors, Medians and Altitudes



(*i*) Length of an angle bisector from the angle $A = \beta_a = \frac{2bc \cos \frac{A}{2}}{b+c}$.

(*ii*) Length of median from angle $A = m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$.

(*iii*) Length of altitude from the angle $A = A_a = \frac{2\Delta}{a}$.

12. The Distances of the special Points from Vertices and Sides of Triangle

(*i*) Circumcentre (O) : $OA = R$ and $O_a = R \cos A$

(*ii*) Incentre (I) : $IA = r \operatorname{cosec} \frac{A}{2}$ and $I_a = r$

(*iii*) Excentre (I_1) : $I_1 A = r_1 \operatorname{cosec} \frac{A}{2}$

(*iv*) Orthocentre : $HA = 2R \cos A$ & $H_a = 2R \cos B \cos C$

(*v*) Centroid (G) : $GA = \frac{1}{3}\sqrt{2b^2 + 2c^2 - a^2}$ and $G_a = \frac{2\Delta}{3a}$

13. Orthocentre and Pedal Triangle

The triangle KLM which is formed by joining the feet of the altitudes is called the Pedal Triangle.

(*i*) Its angles are $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$.

(*ii*) Its sides are $a \cos A = R \sin 2A$,

$b \cos B = R \sin 2B$ and

$c \cos C = R \sin 2C$

(*iii*) Circumradii of the triangles *PBC*, *PCA*, *PAB* and *ABC* are equal.

Where P is orthocenter of ΔABC.

14. Excentral Triangle

The triangle formed by joining the three excentres I_1, I_2 and I_3 of ΔABC is called the excentral or excentric triangle.

(*i*) ΔABC is the pedal triangle of the $\Delta I_1 I_2 I_3$.

(*ii*) Its angles are $\frac{\pi}{2} - \frac{A}{2}, \frac{\pi}{2} - \frac{B}{2}$ and $\frac{\pi}{2} - \frac{C}{2}$.

(*iii*) Its sides are $4R \cos \frac{A}{2}$, $4R \cos \frac{B}{2}$ and $4R \cos \frac{C}{2}$.

(*iv*) $I I_1 = 4R \sin \frac{A}{2}; I I_2 = 4R \sin \frac{B}{2}; I I_3 = 4R \sin \frac{C}{2}$.

(*v*) Incentre I or ΔABC is the orthocentre of the excentral $\Delta I_1 I_2 I_3$.

15. Distance Between Special Points

(*i*) Distance between circumcentre and orthocentre

$OH^2 = R^2 (1 - 8 \cos A \cos B \cos C)$

(*ii*) Distance between circumcentre and incentre

$OI^2 = R^2 \left(1 - 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}\right) = R^2 - 2Rr$

(*iii*) Distance between circumcentre and centroid

$OG^2 = R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$